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THE DEVELOPMENT OF THE FLOW OF VISCOUS AND VISCOPLASTIC MEDIA BETWEEN TWO PARALLEL PLATES†

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The acceleration and the establishment of viscos and viscoelastic media between two parallel plates acted upon by an abruptly applied constant pressure gradient is investigated. The Shvedov–Bingham model is used for the viscoplastic medium and a solution is sought in the form of layered flow with a time-dependent thickness of the solid kernel. The problem is reduced to a one-dimensional heat-conduction equation for a rod of variable length, where the expansion law is not known in advance and an additional non-linear boundary condition is used to determine it. Two versions of the approximate solution are given. The first is a general modified Slezkin–Targ method, by means of which the problem of the motion of the medium acted upon by the specified time-varying pressure gradient is reduced to solving an ordinary differential equation. The second is an asymptotic method of solving the problem for short and long values of the dimensionless time. The time dependences of the flow rate and the boundary of the solid kernel are constructed for various rations of the pressure drop to the shear stress limit. The time taken for the flow rate and the boundary of the solid kernel to become established are determined as function of this ratio. © 2000 Elsevier Science Ltd. All rights reserved.

The unsteady flows of a viscous fluid between two fixed parallel plates has been investigated fairly fully, due to the fact that the problem can be reduced to the classical heat-conduction equation. Exact solutions have been obtained using the Fourier method [1–4] and written in the form of series. The approximate Slezkin–Targ method [5, 6] is also of interest. In this method the acceleration of the fluid is replaced by an average over the cross-section. The solution is simplified considerably but may contain a fairly large error.

Problems of transient viscoplastic flows can be reduced to solving non-linear boundary-value problems, which give rise to serious mathematical difficulties. Using the idea of the Slezkin–Targ method, an approximate solution of the problem of the slowing down of a viscoplastic medium has been presented in [7, 8]. A description of the different approaches in investigating unsteady viscoplastic flows is given in [9]. Exact solutions are usually constructed by the semi-inverse method in which the boundary conditions on the boundary of the solid kernal are chosen from the form of the corresponding analytic solution. As a result, solutions are obtained for boundary-value problems that are unnatural from the point of view of experiment. A wide class of multiparametric accurate solutions of the problem of the flow of a viscoplastic medium acted upon by a specified time-varying pressure gradient has been presented in recent papers [10, 11].

1. FORMULATION OF THE BOUNDARY-VALUE PROBLEM

Consider the flow of a viscoplastic medium in the region $0 \leqslant \bar{z} \leqslant 2h$ between two fixed parallel plates $\bar{z} = 0$, $\bar{z} = 2h$. We will use the Shvedov–Bingham constitutive relation in the form $\tau = \mu\partial\bar{\upsilon}/\partial\bar{z} \pm \tau_0$, where τ – the shear stress on the area $\bar{z} = \text{const}$ – exceeds τ_0 (the shear stress limit) in absolute value. In the region where $|\tau| \leqslant \tau_0$, the deformation rate is zero and, consequently, the velocity υ is independent of the $\bar{z}$ coordinate. This region is called the rigid kernel. The medium is at rest at the initial instant $t = 0$. When $t > 0$ flow begins due to the action of a constant pressure gradient $d\bar{p}/d\bar{x}$, $t > 0$, directed along the $\bar{x}$ axis.

The determination of the velocity field of the medium υ $(\bar{t}, \bar{z})$ reduces to solving the following boundary-value problem [10]

$$\rho\frac{\partial\bar{\upsilon}}{\partial\bar{t}} = \mu\frac{\partial^2\bar{\upsilon}}{\partial\bar{z}^2} - \frac{\partial\bar{p}}{\partial\bar{x}}, \quad 0 \leqslant \bar{z} \leqslant \bar{z}_0(\bar{t})$$

$$\bar{\upsilon}\,(0,\bar{z}_0) = 0, \quad \bar{\upsilon}\,(\bar{t},0) = 0, \quad \left.\frac{\partial\bar{\upsilon}}{\partial\bar{z}}\right|_{\bar{z}=\bar{z}_0} = 0, \quad \mu\left.\frac{\partial^2\bar{\upsilon}}{\partial\bar{z}^2}\right|_{\bar{z}=\bar{z}_0} = -\frac{\tau_0}{h-\bar{z}_0}$$

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In the region $\bar{z} < \bar{z} < h$ the function $\upsilon(\bar{t}, \bar{z})$ is equal to the constant value $\upsilon(\bar{t}, \bar{z}_0)$, $|\tau(\bar{t}, \bar{z})| < \tau_0$. In the region $h < \bar{z} < 2h$ the functions $\upsilon(\bar{t}, \bar{z})$ are supplemented with respect to symmetry $\upsilon(\bar{t}, \bar{z}) = = \upsilon(\bar{t}, 2h - \bar{z})$.

We introduce the dimensionless quantities

$$v = \left(\left|\frac{\partial p}{\partial \bar{x}}\right| \frac{h^2}{\mu}\right)^{-1} v, \quad P = -\frac{h}{\tau_0}\frac{\partial p}{\partial \bar{x}}, \quad z = \frac{\bar{z}}{h,} \quad z_0 = \frac{\bar{z}_0}{h}, \quad t = \frac{\rho h^2}{\mu}\bar{t} \tag{1.1}$$

Then, to determine the dimensionless velocity $\upsilon(t, z)$ and the function $z_0(t)$ from (1.1) and (1.2) we obtain the boundary-value problem

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial z^2} + 1, \quad t \geqslant 0, \quad 0 \leqslant z \leqslant z_0(t) \tag{1.2}$$

$$v(t,0) = 0, \quad v_z(t, z_0) = 0, \quad v_{zz}(t, z_0) = -\frac{1}{P(1 - z_0)} \tag{1.3}$$

$$v(0, z) = 0, \quad z_0(0) = 0 \tag{1.4}$$

2. THE SOLUTION OF THE PROBLEM FOR A VISCOUS FLUID

The viscous-fluid approximation corresponds to the limit $P \to \infty$. Assuming $z_0(t) = 1$ in (1.2) – (1.4) and omitting the last conditions in (1.3) and (1.4), we obtain a linear boundary-value problem for the classical heat-conduction equation.

Using the method of separation of variables [1–3] we can obtain the solution for the velocity field in the form of a Fourier series

$$v(t,z) = z - \frac{1}{2}z^2 - \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin \pi(n-\frac{1}{2})z}{(n-\frac{1}{2})^3}\exp(-\pi^2(n-\tfrac{1}{2})^2 t)$$

Hence, for the dimensionless flow rate we have the expression

$$Q = 2\int_0^1 u dz = \frac{2}{3} - \frac{4}{\pi^4}\sum_{n=1}^{\infty}(n-\tfrac{1}{2})^{-4} e^{-\pi^2(n-1/2)^2 t} \tag{2.1}$$

Series (2.1) converges for all values of t. However, for small values of t it is more convenient to use the identical series.

$$Q = 2t - \frac{8}{3\sqrt{\pi}}t^{3/2} + r(t), \quad r(t) < \frac{16}{3\sqrt{\pi}}t^{3/2}e^{-1/t}$$
$$r(t) = \sum_{n=1}^{\infty}(-1)^{n+1}a_n, \quad a_n = \frac{4}{\sqrt{\pi}}\int_0^t dt'\int_0^{t'} e^{-n^2/x}\frac{dx}{\sqrt{x}} < \frac{16}{3\sqrt{\pi}}t^{3/2}e^{-n^2/t} \tag{2.2}$$

Series (2.2) can be derived using Poisson's formula [12]

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x - 2n\pi a x} = x^{-1/2} e^{\pi a^2 x}\left(1 + 2\sum_{n=1}^{\infty} e^{-n^2\pi/x}\cos n\pi 2a\right) \tag{2.3}$$

as follows. We differentiate series (2.1) twice with respect to t, we assume $\pi t = x$ on the right-hand side and we convert the sum using Poisson's formula (2.3) with $a = -1/2$. Returning to the initial argument $t = x/\pi$, we obtain

$$Q''(t) = -\frac{2}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-n^2/t}\right) \tag{2.4}$$

We assume $t = 0$ in series (2.1) for $Q(t)$ and its derivative $Q'(t)$. Using the well-known values for the numerical series, we obtain $Q(0) = 0$ and $Q'(0) = 2$. Hence, integrating series (2.4) twice we obtain series (2.2).

For sufficiently small t, series (2.2) converges much more rapidly than (2.1). The order of smallness of the residual term $r(t)$ in (2.2) is higher than any degree of t. The function $r(t)$ is non-analytic in the neighbourhood of $t = 0$ and is represented by a converging sign-variable series, the residual term in which is less than the first neglected term. The first term of the expansion in (2.2), linear in t, is easily obtained from the asymptotic solution (2.1). However, the second term of the expansion of $t^{3/2}$ and, all the more, the last non-analytic terms, can only be obtained using special transformations (see Section 4).

We will demonstrate the effectiveness of series (2.1) and (2.2) for calculating the flow rate $Q(t)$. We will confine ourselves to two terms in the partial sums $Q_+(t)$ of series (2.1) and $Q_-(t)$ of series (2.2)

$$Q_+(t) = 2/3 - 0{,}657e^{-2{,}47t}, \quad Q_-(t) = 2t - 1{,}504t^{3/2}$$

which give upper and lower estimates, respectively, for Q(t). The flow rate can then be determined from the formula

$$Q = \begin{cases} 2t - 1{,}504t^{3/2}, & t \leq t_0 \\ 2/3 - 0{,}657e^{-2{,}47t}, & t > t_0 \end{cases} \tag{2.5}$$

In Fig. 1 we show the functions $r_+(t) = Q_+(t) - Q(t) \approx 0.00811e^{-22.2t}$, $r_-(t) = Q(t) - Q_-(t) \approx 3.01t^{3/2}e^{-1/t}$ which determine the errors of the two-term partial sums (2.5). The greatest error of expression (2.5) arises at the point $t_0 \approx 0.153$ and is equal to $r_+(t) = r_-(t) \approx 2.7 \times 10^{-4}$, while the relative error is less than 0.13%.

3. THE SOLUTION BY AVERAGING THE ACCELERATION

The Slezkin–Targ method was proposed in [5, 6] for calculating the flow of a viscous fluid and it was also used to solve the problem of the impact of a viscoplastic rod [7, 8].

Below we describe a modification of the Slezkin–Targ method, which differs from the traditional method in that a general ordinary differential equation for the flow rate will be obtained. The time dependence of the flow rate can be determined without a detailed calculation of the velocity field, and this can be sufficient to solve a number of practical problems.

We will first illustrate the method and its effectiveness by solving problem (1.4) – (1.6) for a viscous fluid. The acceleration is replaced by an average over the cross-section; hence

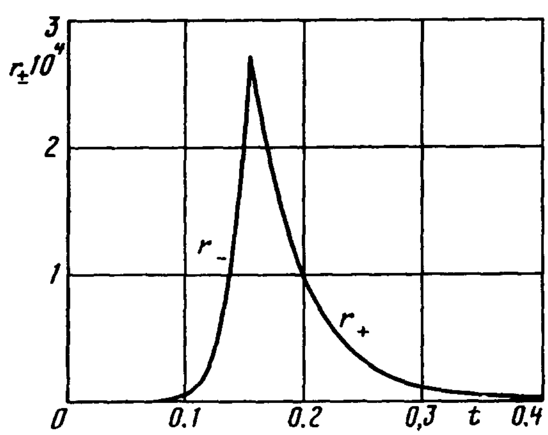

Fig. 1.

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{dQ}{dt} \tag{3.1}$$

Then

$$v(z) = \left(1 - \frac{1}{2}\frac{dQ}{dt}\right)\left(z - \frac{1}{2}z^2\right)$$

Integrating, we obtain the required equation for the flow rate

$$Q(t) = \int_0^2 v(z)dz = \frac{2}{3}\left(1 - \frac{1}{2}\frac{dQ}{dt}\right)$$

which has the following solution, satisfying the condition $Q(0) = 0$

$$Q(t) = \frac{2}{3}(1 - e^{-3t}) \tag{3.2}$$

The principal asymptotic form for $2t$ when $t \ll 1$ is the same as the exact solution, and when $t \gg 1$ the coefficient of the exponential asymptotic form differs from the exact one by 1.5%. The largest difference between solutions (3.2) and (2.1) is 0.044 when $t = t_0$, which is 13.5% of the exact value.

We will now solve boundary-value problem (1.2) – (1.4) by the Slezkin–Targ method for a viscoplastic medium. We replace the acceleration in (1.2) by the average (3.1). The solution of Eq.(1.2), which satisfies the first two conditions (1.3), then takes the form

$$v = \left(1 - \frac{1}{2}\frac{dQ}{dt}\right)\left(z_0 z - \frac{1}{2}z^2\right) \tag{3.3}$$

Substituting (3.3) into the third boundary condition (1.3), we obtain the first equation for the functions $z_0(t)$ and $Q(t)$

$$1 - \frac{1}{2}\frac{dQ}{dt} = \frac{1}{P(1 - z_0)} \tag{3.4}$$

Integrating (3.3) with respect to z, we obtain the second equation for $z_0(t)$ and $Q(t)$

$$Q = 2\int_0^{z_0} v dz + 2(1 - z_0)v(z_0) = \left(1 - \frac{1}{2}\frac{dQ}{dt}\right)\left(z_0^2 - \frac{1}{3}z_0^3\right) \tag{3.5}$$

It is more convenient to write the system of equations (3.4), (3.5) in the form of a differential equation for the variable $\Pi(z_0) = 1/(1 - z_0)$

$$\frac{1}{3}\left(1 - \frac{1}{\Pi^3}\right)\frac{d\Pi}{dt} + \Pi = P, \quad \Pi(0) = 1 \tag{3.6}$$

the solution of which has the form

$$3t = \left(1 - \frac{1}{P^3}\right)\ln\frac{P-1}{P-\Pi} - \frac{1}{P^3}\ln\Pi + \frac{1-\Pi^2}{2P\Pi^2} + \frac{1-\Pi}{P^2\Pi} \tag{3.7}$$

The flow rate can be expressed algebraically in terms of the function $\Pi(t)$

$$Q = \frac{(2\Pi + 1)(\Pi - 1)^2}{3\Pi^2 P} \tag{3.8}$$

It follows from (3.6) and (3.8) that as $t \to \infty$

$$\Pi \to P, \quad Q \to Q_\infty = \frac{(2P+1)(P-1)^2}{3P^3} \tag{3.9}$$

We will now obtain the asymptotic expansion of the exact solution of boundary-value problem (1.2) – (1.4) when $t \ll 1$ and we will compare it with the approximate solution (3.7), (3.8).

4. THE ASYMPTOTIC SOLUTION FOR SHORT TIMES

We will seek the solution in the form of the following expansion

$$v = tF(\xi) + t^{3/2}F_1(\xi) + \ldots, \quad z_0 = 2\sqrt{t}(A + B\sqrt{t} + \ldots), \quad \xi = z/(2\sqrt{t}) \tag{4.1}$$

Hence, we obtain values of the function v and its derivatives with respect to t and z, substituting which into Eq.(1.2) and boundary conditions (1.3), we obtain the following boundary-value problems for determining the functions $F(\xi)$, $F_1(\xi)$ and the constants A and B

$$F''(\xi) + 2\xi F'(\xi) - 4F(\xi) + 4 = 0 \tag{4.2}$$

$$F(0) = 0, \quad F'(A) = 0, \quad F''(A) = -4/P \tag{4.3}$$

$$F_1''(\xi) + 2\xi F_1'(\xi) - 6F_1(\xi) = 0 \tag{4.4}$$

$$F_1(0) = 0, \quad F_1'(A) - 4B/P = 0, \quad F_1''(A) + 8BA/P = -8A/P \tag{4.5}$$

(the last conditions are simplified using the relations $F''(A) = -4$, $F'''(A) = 8A$).

Equation (4.2) has the general solution

$$F(\xi) = c_1 y_1(\xi) + c_2 y_2(\xi) + 1 \tag{4.6}$$

$$y_1(\xi) = 2\xi^2 + 1, \quad y_2(\xi) = (\xi^2 + 1/2)\sqrt{\pi}\,\mathrm{erf}\,\xi + \xi\exp(-\xi^2) \tag{4.7}$$

$$\mathrm{erf}\,\xi = \frac{2}{\sqrt{\pi}}\int_0^\xi e^{-x^2}dx$$

where $y_1(\xi)$ and $y_2(\xi)$ – are independent solutions of homogeneous equation (4.2)

For the constants c_1, c_2 and A we obtain from (4.3)

$$c_1 = -1, \quad c_2 = 2A\exp(A^2)/P(A) \tag{4.8}$$

$$P(A) = 1 + \sqrt{\pi}A\exp(A^2)\,\mathrm{erf}\,A =$$

$$= \begin{cases} 1 + \alpha_1 A^2 + \alpha_2 A^4 + \alpha_3 A^6 + \ldots + \alpha_n A^{2n} + \ldots \\ \sqrt{\pi}Ae^{A^2} + (\alpha_1 A^2)^{-1} - (\alpha_2 A^4)^{-1} \ldots + (-1)^n(\alpha_n A^{2n})^{-1} + \ldots \end{cases} \tag{4.9}$$

$$(\alpha_n = 2^n/(2n-1)!!, \quad \alpha_1 = 2, \quad \alpha_2 = 4/3, \ldots)$$

The last expression in (4.8) is a parametrization of the dimensionless number $P(A)$, which occurs in boundary-value problem (1.2) – (1.4). The first series of (4.9) converges for any A, while the second series, which is asymptotic to it, is convenient to use for calculating the function $P(A)$ for large A. The residual term for the second series does not exceed the first neglected term.

We can similarly obtain the solution of problem (4.4), (4.5). Equation (4.4) has two independent solutions

$$Y_1 = 2\xi^3 + 3\xi, \quad Y_2(\xi) = Y_1(\xi)(\sqrt{\pi}/2)\,\mathrm{erf}\,\xi + (\xi^2 + 1)\exp(\xi^2) \tag{4.10}$$

The family of solutions $F_1(\xi) = C(2\xi^3 + 3\xi)$ satisfies the first condition of (4.5). Substituting this expression into the second two conditions of (4.5) we obtain a system for determining the constants B

and C. Finally, using relations (4.6) – (4.8) we obtain $F(\xi)$, and from the system mentioned we obtain the constants B and C and the function $F_1(\xi)$

$$F(\xi) = -2\xi^2 + 2(A / P(A))\exp(A^2)((\xi^2 + \tfrac{1}{2})\sqrt{\pi}\,\mathrm{erf}\,\xi + \xi\exp(-\xi^2))$$
$$F_1(\xi) = -\frac{4(2\xi^2+3)\xi}{3P(2A^2+3)} \tag{4.11}$$

The constant A is obtained from the equation $P(A) = P$ and the function $P(A)$ is defined by series (4.9).

The velocity is found from expansion (4.1). In particular, the velocity of the solid kernel

$$v(t, z_0) = tF(A) + t^{3/2}\sqrt{t}F_1(A) + \ldots = t(1 - 1/P) - t^{3/2}4A/(3p) + \ldots \tag{4.12}$$

The flow rate in the solid kernel is equal to the product of the velocity (4.12) and the width of the kernel $2(1 - z_0)$. The flow rate outside the kernel is given by the integral

$$2\int_0^{z_0} v(t, z)dz = 4t^{3/2}\int_0^A F(x)dx = t^{3/2}\left(4A - \frac{8A}{3P}\exp(A^2) - \frac{4A}{3P}\right) \tag{4.13}$$

The integral in (4.13) is calculated using the equation

$$-F'(0) + 2AF(A) - 6\int_0^A F(\xi)d\xi + 4A = 0$$
$$F(A) = 1 - 1/P, \quad F'(A) = 0, \quad F'(0) = 4A\exp(A^2)/P \tag{4.14}$$

which is obtained by integrating Eq.(4.2) with respect to ξ in the limits from 0 to A.

For the total flow rate at the initial stage $t \ll 1$ we have

$$Q = 2t\left(1 - \frac{1}{P}\right) - t^{3/2}\frac{8A}{3P}\exp(A^2) + \ldots \tag{4.15}$$

5. THE ASYMPTOTIC SOLUTION FOR LONG TIMES

We will seek the solution of boundary-value problem (1.2) – (1.4) in the form

$$v = v_\infty(z) - \varepsilon(z)\exp(-\alpha^2 t), \quad z_0 = z_\infty - \beta\exp(-\alpha^2 t) \tag{5.1}$$

$$v_\infty(z) = z_\infty z - \tfrac{1}{2}z^2, \quad z_\infty = 1 - 1/P \tag{5.2}$$

where $v_\infty(z)$, z_∞ – are the velocity and the boundary of the solid kernel of the steady flow respectively as $t \to \infty$.

We substitute expressions (5.1) into (1.2) and (1.3) and we take into account only terms linear in ε and β. We then obtain the boundary-value problem for determining the function $\varepsilon(t)$ and the constant β

$$\varepsilon''(z) + \alpha^2\varepsilon(z) = 0 \tag{5.3}$$

$$\varepsilon(0) = 0, \quad \varepsilon'(z_\infty) = \beta, \quad \varepsilon''(z_\infty) = -\beta P \tag{5.4}$$

To determine the function $\varepsilon(z)$ we must eliminate β from (5.4), after which the solution of the boundary-value problem can be represented in the form of a linear combination of the eigenfunctions $\varepsilon_n \sin(\alpha_n z)$. The eigenvalues α_n are found from the algebraic equation

$$\frac{\alpha_n}{P}\mathrm{tg}\left(\frac{P-1}{P}\alpha_n\right) = 1 \tag{5.5}$$

The following expansions hold for the smallest positive root α_0 of Eq.(5.5)

$$\alpha_0 = \begin{cases} P(P-1)^{-1/2}\left(1-\frac{1}{6}(P-1)+\frac{11}{360}(P-1)^2\right), & P \leq 2 \\ \frac{\pi}{2}+\frac{1}{3}\left(\frac{\pi}{2P}\right)^3-\left(\frac{1}{5}-\frac{1}{3P}\right)\left(\frac{\pi}{2P}\right)^5, & P>2 \end{cases} \tag{5.6}$$

the relative error of which is less than 1%.

Each successive root α_n exceeds the previous root α_{n-1} by more than $\pi P/(P-1)$ and lies in the range $\pi n P/(P-1) < \alpha_n < (\pi/2 + \pi n)P/(P-1)$. The smallest eigenvalue $\alpha = \alpha_0$ defines the asymptotically principal terms in (5.1). Using (5.2) and (5.4) for these we obtain

$$\varepsilon(z) = \varepsilon_0 \sin \alpha_0 z, \quad \beta = \varepsilon'(z_\infty) = \varepsilon_0 \alpha_0 \cos[(P-1)\alpha_0 / P] \tag{5.7}$$

We determine the flow rate

$$Q = 2(1-z_0)v(t, z_0) + 2\int_0^{z_0} v\, dz$$

Substituting expressions (5.1) and (5.7) here and taking into account terms that are linear in ε and β, we obtain the following expansion for the flow rate at the relaxation stage ($t \gg 1$)

$$Q_+ = Q_\infty(P) - \lambda(P)\exp(-\alpha_0^2 t), \quad Q_\infty(P) = \frac{(2P+1)(P-1)^2}{3P^3} \tag{5.8}$$

$$\beta = \frac{\lambda\alpha_0^2}{4P}, \quad \varepsilon_0 = \frac{\lambda P}{4\sin[(P-1)\alpha_0 / P]} \tag{5.9}$$

where Q_∞ is the limit value of the flow rate as $t \to \infty$ (see (3.9)).

6. COMBINED EXPANSION

The number $\lambda(P)$ is found from the condition for smooth matching of the functions Q_+(5.8) and Q_-(4.14) at the point t_0

$$Q_+(t_0) = Q_-(t_0), \quad Q_+'(t_0) = Q_-'(t_0) \tag{6.1}$$

We initially eliminate λ from (6.1)

$$\alpha_0^2 Q_+(t_0) + Q_+'(t_0) = \alpha_0^2 Q_-(t_0) + Q_-'(t_0) \tag{6.2}$$

Substituting (5.8) and (4.14) into (6.2), we obtain the equation

$$t_0^{1/2} - pt_0 + qt_0^{3/2} = r \tag{6.3}$$

in which p, q and r are functions of the parameter P. After calculating $t_0(P)$ we can find λ from (6.1)

$$\lambda = \exp(\alpha_0^2 t_0)(Q_\infty - Q_-(t_0)) \tag{6.4}$$

The quantity t_0 defines the limit of the action of the two asymptotic forms $Q_+(t)$ and $Q_-(t)$. The error in determining it has only a minor effect on the value of λ in (6.4). Hence, it is sufficient to confine ourselves to a simple interpolation for the function $t_0(P)$. When $P \gg 1$ the asymptotic forms $Q_+(t)$ and $Q_-(t)$ convert to the asymptotic forms (2.5) for a viscous liquid for which $t_0 \approx 0.15$. When $P-1 \ll 1$ the following asymptotic forms hold for the coefficients of Eq.(6.3) and t_0

$$p \approx \frac{1}{\sqrt{2(P-1)}}, \quad q \approx \frac{2}{3(P-1)}, \quad r \approx \sqrt{\frac{P-1}{8}}, \quad t_0 \approx 0{,}186(P-1) \tag{6.5}$$

We can propose the following interpolation for t_0.

$$t_0 = \frac{P-1}{6{,}5P-1{,}2} \tag{6.6}$$

which takes into account both limiting cases $P \gg 1$ and $P - 1 \ll 1$.

Formulae (6.4) and (6.5) define the lacking parameter λ in the expression for the flow rate. The constants β and ε_0 are expressed in terms of λ using formulae (5.9), and then, using (5.1) we find the velocity field and the boundary of the solid kernel $z_0(t)$. Finally, we obtain the following formulae for the flow rate Q and the boundary of the kernel $z_0(t)$

$$\begin{aligned} Q &= Q_-(t, P), \quad z_0 = z_-(t, P), \quad t < t_0(P) \\ Q &= Q_+(t, P), \quad z_0 = z_+(t, P), \quad t > t_0(P) \end{aligned} \tag{6.7}$$

The function $Q_-(t, P)$, is defined by (4.14), while $Q_+(t, P)$ is found from relations (5.8), (6.4) and (6.6). The function $z_-(t, P)$, can be obtained from relation (4.1), while $z_+(t, P)$ can be found from (5.1) using (5.2) and (5.9)

$$z_-(t, P) = 2A\sqrt{t} - 2\frac{1+2A^2}{3+2A^2}t \tag{6.8}$$

$$z_+(t, P) = 1 - \frac{1}{P} - \frac{\lambda\alpha_0^2}{4P}\exp(-\alpha_0^2 t) \tag{6.9}$$

We will now analyse the solutions obtained.

In the limiting case $P - 1 \ll 1$ (flow close to purely plastic flow) we can use the asymptotic forms (6.5) and

$$Q_\infty \approx (P-1)^2, \quad \alpha_0^2 \approx 1/(P-1), \quad A = \sqrt{(P-1)/2}$$

$$Q_-(t_0) \approx 0{,}221(P-1)^2, \quad \lambda = 0{,}938(P-1)^2$$

Hence, we obtain from (4.15) and (6.7)–(6.9)

$$\begin{aligned} Q_- &= 2(P-1)t - \tfrac{4}{3}\sqrt{2(P-1)}t^{3/2} \\ z_- &= \sqrt{2(P-1)t} - \tfrac{2}{3}t \\ Q_+ &= (P-1)^2(1 - 0{,}938\exp(-t/(P-1))) \\ z_+ &= (P-1)(1 - 0{,}234\exp(-t/(P-1))) \end{aligned} \tag{6.10}$$

In the opposite limiting case $P \gg 1$ (viscous flow) using the asymptotic forms

$$Q_\infty \approx \tfrac{2}{3}; \quad \alpha_0^2 \approx 2{,}467; \quad A\exp(A^2) \approx P/\sqrt{\pi};$$

$$Q_-(t_0) \approx 0{,}216, \quad \lambda \approx 0{,}657, \quad \beta \approx 0{,}405/P, \quad t_0 = 0{,}153$$

for the flow rate we obtain expressions identical with (2.7) – (2.9) for a viscous liquid, and for the boundary of the solid kernel we have

$$z_- = 2A\sqrt{t} - 2t, \quad z_+ = 1 - 1/P - 0{,}405e^{-2{,}47t}/P \tag{6.11}$$

In Fig. 2 we show curves of $Q(t)/Q_\infty$ for three different values of A, the values of which are indicated ($A = 0.3$, $P(A) = 1.19$, $A = 0.5$, $P(A) = 1.59$ and $A = 1.2$, $P(A) = 9.17$). The continuous curves are calculated using the asymptotic formulae (6.7), (4.15) and (5.8), and the dashed curves are calculated using the approximate solution (3.7) and (3.8) obtained by the Slezkin–Targ method.

The curve of $Q(t)/Q_\infty$ for $A = 1.2$ differs only slightly from the exact curve for a viscous liquid (2.7)–(2.9). The largest difference between the approximate solution (3.7) and (3.8) and the exact solution for a viscous liquid ($P \gg 1$) is 13.5% and occurs when $t = 0.153$. The relative error decreases when the

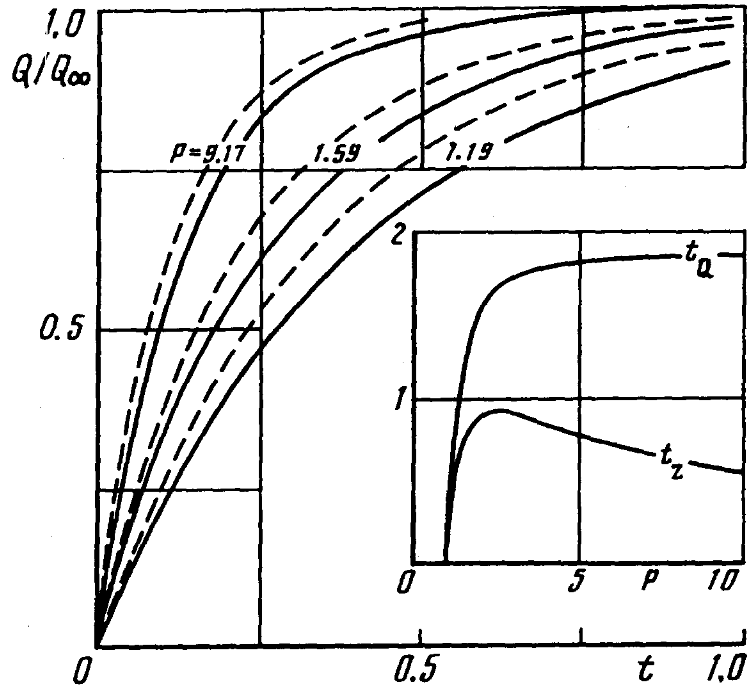

Fig. 2.

parameter P decreases. Hence, the relative error in determining the flow rate using the Slezkin–Targ method does not exceed 13.5%.

The dependence on P of the relaxation time t_Q for the flow rate and t_z for the boundary of the solid kernel is determined from the condition that the corresponding value differs from its limit value by 1%. Using asymptotic relations (5.9) and (6.9) we obtain

$$t_Q = \frac{1}{\alpha_0^2}\ln\frac{100\lambda}{Q_\infty}, \quad t_z = \frac{1}{\alpha_0^2}\ln\frac{25\lambda\alpha_0}{P-1}$$

When $P - 1 \ll 1$ we obtain from (6.10) $t_Q \approx 4.54(P-1)$, $t_z \approx 3.15(P-1)$. When $P \gg 1$ we obtain from (2.7) and (6.11) $t_Q \approx 1.86$ and $t_z \approx 1.5 - 0.405 \ln P$.

In Fig. 2 we show curves of $t_Q(P)$ and $t_z(P)$. The function $t_Q(P)$ increases monotonically. The function $t_z(P)$ increases along the section $0 \leqslant P < 2.4$, when $P \approx 2.4$ it reaches its greatest value $t_z \approx 0.93$, and when $P > 2.4$ it slowly decreases. When $P \approx 28$ we have $t_z \approx t_0$. When $P > 28$ the time taken for the boundary of the solid kernel to become established is less than t_0, and it must be found from (6.11) for the inner asymptotic form $z_-(t)$, whence

$$t_z \approx \frac{1}{4A^2} \approx \frac{1}{4\ln P}$$

Hence, for a large value of P the boundary of the solid kernel $z_0 = 1 - 1/P$ is established very rapidly and a longer process of establishing the velocity and the flow rate then occurs.

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